Practically Oriented Finite-Time Control Design and Implementation: Application to Series Elastic Actuator

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Abstract—This paper proposes a practically oriented finite-time control design for nonlinear systems under a less ambitious but more practical semi-global control objective. As one main contribution, the designed finite-time controller can be expressed as a very simple form while the control gain tuning mechanism follows a conventional pole placement manner, which enhances its facility in practical implementations. Moreover, by utilizing a modified non-recursive homogeneous domination design without pre-verifying any nonlinearity growth constraints, the control scheme can be directly obtained by totally neglecting the recursive calculations of series virtual controllers. Rigorous semi-global attractiveness and local finite-time convergence analysis are presented to ensure the theoretical justification. A control application and experimental verification to Series Elastic Actuator demonstrate the control effectiveness and significant performance improvements compared with asymptotical state feedback controllers.

Index Terms—finite-time control design, robust control, semi-global stability, series elastic actuator

I. INTRODUCTION

Finite-time control issue for nonlinear systems has been extensively studied in the last decades owing to its well known fast convergence rate and stronger robustness against system uncertainties/disturbances. A pioneer significant theoretical contribution for continuous finite-time control can be referred to the research of double integrators in [1] where a homogeneous second-order controller is proposed to achieve a global finite-time stability. Later research interests mainly focus on the nonsmooth extension of recursive backstepping design to realize finite-time stability of $n$-th order control systems with the presence of nonlinear perturbations. Notably, an effective nonsmooth design approach, namely, adding a power integrator, is first proposed in [2], [3]. Meanwhile, an extended backstepping design strategy is also proposed in [4] to solve the finite-time stabilization problem for a class of perturbed chain of power integrators. With a few years’ development, the nonsmooth control design method is greatly enriched by taking fully advantage of the weighted homogeneity [5], [6]. It is later reported in [7], [8] that a homogeneous domination design methodology can be systematically addressed to solve the finite-time output feedback stabilization problem for a class inherent nonlinear systems. From a practical point of view, finite-time control design for practical systems have also been extensively studied, see e.g., [9]–[11], etc. On the other hand, finite-time control via an alternative non-singular terminal sliding mode can also be widely found in the literature, for examples [12], [13], only mention a few.

However, it is well known that a common nonlinear growth constraint of the considered system is essentially required for global continuous finite-time stabilization design, which is described detailedly in [8]. For general real-life plants, it is clearly of suspicion that most of the nonlinear hypothesis could be widely satisfied. Hence, a key question arises that for a general class of nonlinear systems of the form (1), how to essentially relax the pre-requirement of the system nonlinearity constraints, such that a practically oriented finite-time controller can be widely applied. On the other hand, due to massive utilization of recursive domination approaches, the obtained guideline of the control gains are usually very conservative and sometimes even the control gains are required to be smooth functions of the system state, see for instances [3], [14], etc. Typically, the conservative feature will be much severe when handling high-order systems. In practices, this feature will add much complexity for implementations and might cause an apparent performance deterioration when system is suffering from measurement noises.

In this paper, inspired by a recent advance on nonsmooth stabilization design [15], we propose a practically oriented finite-time control design strategy for system (1) under a less ambitious but more practical control objective, namely, semi-global rather than restrictive global control. Firstly, a delicate coordinates transform is presented by exactly calculating the steady-state generators. Secondly, we show by a non-recursive design that the controller scheme design can be essentially separated with its stability analysis, which is inevitable in existing recursive design approaches. Thirdly, a rigorous semi-global attractiveness and local finite-time convergence provide the theoretical justifications of the proposed method. Compared with existing related finite-time control design results for lower-triangular nonlinear systems, the main distinguishable improvements are stated as follows. First, under a semi-global control infrastructure, we show that existing essentially
required nonlinearity growth constraints, which are restrictive to be verified, can now be fully removed. Second, the involved tuning homogeneous degree, which could significantly affect the control performance, has been endowed with much flexibility within a tunable region. Third, the proposed novel non-recursive synthesis approach could render a finite-time control scheme to be designed in a very simple expression, whereas its control gain selection follows a conventional pole placement manner. More specifically, the obtained controller can reduce to its linear state feedback controller counterpart simply by assigning the homogeneous degree to zero.

At the end, an application and experimental verification results to a Series Elastic Actuator (SEA) are included in order to illustrate the simplicity and effectiveness of the proposed control design strategy. Moreover, to provide better understanding of the gain tuning mechanism, a detailed parameter configuration procedure is presented. Then the detailed experimental comparison results with a conventional PD, a linear state feedback controller and an optimal controller are provided as well to demonstrate the control performance improvements.

Notations: For integers $i$ and $j$ satisfying $0 \leq j \leq i$, denote $\mathbb{N}_{i,j} := \{j, j+1, \ldots, i\}$. The symbol $\mathbb{C}^i$ denotes the set of all differentiable functions whose first $i$th time derivatives are continuous. A continuous function $\lfloor \cdot \rceil^a$ is defined by $\lfloor \cdot \rceil^a = \text{sign}(\cdot) \lfloor \cdot \rceil^a$ where $a \in \mathbb{R}_+$ is a constant.

II. PRELIMINARIES AND PROBLEM STATEMENT

A. Preliminaries

The following notations are provided for brevity of expressions.

i) (Weighted Homogeneity) [6] For a fixed choice of coordinates $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, and positive real numbers $(r_1, r_2, \cdots, r_n) \equiv r$, a one-parameter family of dilation is a map $\Delta_r : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, defined by $\Delta_r x = (e^{r_1 x_1}, \cdots, e^{r_n x_n})$. For a given dilation $\Delta_r$ and a real number $\tau$, a continuous function $V : \mathbb{R}^n \to \mathbb{R}$ is called $\Delta_r-$homogeneous of degree $\tau$, denoted by $V \in H^\tau_{\Delta_r}$, if $V \circ \Delta_r = e^{\tau V}$. A continuous vector field $f(x) = \sum f_j(x) (\frac{\partial}{\partial x_j})$ is $\Delta_r-$homogeneous of degree $\tau$, if $f_j \in H^\tau_{\Delta_r}$, $j \in \mathbb{N}_{1,n}$. Through out this paper, $r$ is given by $r_1 = 1, r_i = r_{i-1} + \tau + 1 = (i+1)\tau, \forall i \in \mathbb{N}_{2,n}$ with a degree $\tau \in (-\frac{1}{n}, 0)$. A homogeneous vector $x_N^\tau$ is denoted by $x_N^\tau = (x_1^\tau, \cdots, x_N^\tau)^T$, $x_N^\tau = x_N$ and $\|x\|^\tau_{\Delta_1} = (\|x_1\|^\tau, \cdots, \|x_N\|^\tau)^T$, $\|x\|^\tau_{\Delta_r} = (\sum_{i=1}^n \|x_i\|^\tau_{\Delta_r}^{r_i})^{1/r}$ denotes a homogeneous p-norm and $\|x\|_{C^s} = (\sum_{i=1}^n \|x_i\|_s^{p_i})^{1/p_i}$ denotes a conventional $L_p$ norm. In this paper, we choose $p = 2$ for simplicity.

ii) [16] Consider a system $x = f(x) + g(x)u$, $x \in \mathbb{R}^n$, with $x = 0$ as its equilibrium. This system is said to be semi-globally stabilizable if, for each (arbitrarily large) compact subset $\mathbb{P} \subset \mathbb{R}^n$, there exists a feedback law $u = u(x)$, which in general depends on $\mathbb{P}$, such that the equilibrium is locally asymptotically stable and $x(t_0) \in \mathbb{P} \Rightarrow \lim_{t \to \infty} x(t) = 0$.

B. Problem Statement

In this paper, we revisit the finite-time control problem for a class of lower-triangular nonlinear systems of the form [3],

$$\begin{align*}
\dot{x}_i(t) &= x_{i+1}(t) + \phi_i(x_i(t)), \quad i \in \mathbb{N}_{1,n-1}, \\
x_n(t) &= u(t) + \phi_n(x_n(t)), \\
y(t) &= x_1(t),
\end{align*}$$

(1)

where $x_i = \text{col}(x_1, \cdots, x_i), i \in \mathbb{N}_{1,n-1}$ and $x = \bar{x}_n$ are the system partial and full state vector, respectively; $y$ is the system output and $\phi_i(\cdot), i \in \mathbb{N}_{1,n}$ is a known smooth nonlinear function. The output reference signal, denoted by $y_r$, and its $n-$th order derivative are assumed to be piecewise continuous, known and bounded.

To achieve a realizable control scheme, firstly, we define $\bar{x}^*_i = \text{col}(x^*_1, \cdots, x^*_i), i \in \mathbb{N}_{1,n-1}$ as an auxiliary variable vector where $x^*_i$ is determined by the following steady-state generators

$$\begin{align*}
x^*_1 &= y_r, \\
x^*_i &= \frac{dx^*_i}{dt} - \phi_i(x^*_i), \quad i \in \mathbb{N}_{2,n-1}.
\end{align*}$$

(2)

Secondly, denote $z = \text{col}(z_1, \cdots, z_n)$ where $z_i = (x_i - x^*_i)/\ell^{i-1}, i \in \mathbb{N}_{1,n}$, $v = (u - x^*_n)/\ell^n$ and $\ell \geq 1$ is a bandwidth factor to be determined later in the stability analysis. In this paper, we show that, without going through the procedure of recursive stability analysis, a simple finite-time controller can be explicitly pre-built of the following form

$$\begin{align*}
v &= -k_1 z_1^{\lfloor 1\rceil^a} - k_2 z_2^{\lfloor 1\rceil^a} - \cdots - k_n z_n^{\lfloor 1\rceil^a} \\
&\quad \pm K [\bar{z}^\tau_{\Delta_1}], \\
u &= \ell^n v + x^*_{n+1},
\end{align*}$$

(3)

with $K = [k_1, \cdots, k_n]$ is the coefficient vector of a Hurwitz polynomial $p(s) = s^n + k_n s^{n-1} + \cdots + k_2 s + k_1$ and the dilation weight is defined by $r = (1, 1 + \tau, \cdots, 1 + (n-1)\tau)$ with a homogeneous degree $\tau \in (-\frac{1}{n}, 0)$.

Remark 2.1: Nonlinear synthesis approaches with simple controller expression, less calculation tasks and easier gain tuning mechanism are always imperatively demanded in practices. In this paper, a finite-time control method can now be easily implemented without much additional complexities comparing to its linear state feedback control counterpart, as sketched by Fig. 1. Comparing with existing related nonlinear control results, such as [18]-[20], etc., the tracking control law can now be designed in a very simple manner of the form (3). It is worthy pointing out that by setting the homogeneous $\tau = 0$, the proposed controller reduces to a conventional linear state feedback control law, i.e., $u = -\ell^n K z + x^*_{n+1}$.

Fig. 1: The block diagram of the proposed finite-time control method.
Remark 2.2: The proposed regulation methodology presents us an alternative, but more practical control design procedure owing to the semi-global control infrastructure. More distinguishably, the common requirements of various nonlinearity growth conditions on the system nonlinearities which are always employed in existing finite-time control related literatures such as [3], [21]–[23], are essentially relaxed. Hence any nonlinear systems of the form (1) can be finite-time controlled by the proposed controller (3) where the main difference relies on the determine guideline of the bandwidth factor $\ell$.

III. MAIN RESULT

The main result of this paper is presented by the following theorem.

Theorem 3.1: For any given constant $\rho \in \mathbb{R}^+$ which could be arbitrary large, consider the closed-loop system consisting of (1) satisfying $x(0) \in \mathcal{U}_x \triangleq [-\rho, \rho]^n$ and the finite-time control law (3) with a sufficiently large bandwidth factor $\ell$ which is dependent on $\rho$. Then the following statements hold.

- Any trajectory starting from the compact set $\mathcal{U}_x$ will converge to its equilibrium $x = x^*$.
- There exists a finite time instant $T > 0$ such that $\forall t \geq T$, $y(t) = y_r$.

Proof: Based on the relation (2), the $z$-system can be directly expressed as

$$
\begin{aligned}
\dot{z}_i &= \xi z_{i+1} + (\phi_i(x_i) - \phi_i(x^*_i))/\ell^{i-1}, \quad i \in \mathbb{N}_{1:n-1}, \\
\dot{z}_n &= \ell v + (\phi_n(x) - \phi_n(x^*_n))/\ell^{n-1}.
\end{aligned}
$$

(4)

Motivated by [6] and [24], construct a simplified Lyapunov function $V(z)$ of the following form

$$
V(z) = \left(\frac{z}{\lambda V(z)}\right)^T P \frac{z}{\lambda V(z)},
$$

(5)

where $P$ is a positive definite and symmetrical matrix satisfying $\Lambda^T P + PA = -I$ with $\Lambda$ being a companion matrix of $K$. From the definition of $K$, it is obvious that $\Lambda$ is Hurwitz.

To simplify the proof, the following proposition whose proof is collected in the Appendix is given first.

Proposition 3.1: The following statements hold.

- $V(z) \in \mathcal{C}^1 \cap \mathcal{B}_{\lambda V(z)}$.
- There exist constants $\varepsilon \in (0, 1/n)$ and $\alpha \in \mathbb{R}_+$ such that
  $$
  \frac{\partial V(z)}{\partial z} \left(\begin{array}{c}
z_2, \cdots, z_n, -\Lambda z_{\lambda V(z)}\end{array}\right)^T \leq -\alpha \hat{V}^{\frac{\tau}{2}}(z), \quad \tau \in (-\varepsilon, 0).
  $$

From the definition of $x_i^*$ and $y_r$, there exists a constant $\bar{\rho} > 0$ such that $\max_{x \in \Omega, y \in \Omega} [\sup_{t \in [0, \tau]} |x^*_i(t)|] \leq \bar{\rho}$. Then for given compact set $\mathcal{U}_x$, define a level set $\mathcal{U}_x = \left\{ z \in \mathbb{R}^n | V(z) \leq \sup_{z \in \Omega} V(z) \right\}$. Let $N = \sup_{z \in \Omega} \| z \|_{\infty}$ where $\| z \|_{\infty}$ stands for the $\mathcal{B}$ norm of vectors, $\mathcal{U}_N = [-N, N]^n$. It is not difficult to conclude that $\mathcal{U}_x \subseteq \mathcal{U}_r \subseteq \Omega \subseteq \mathcal{U}_N$.

With Proposition 3.1 in mind, the time derivative of $V(z)$ along the closed-loop system (3)-(4) gives

$$
\dot{V}(z) = \frac{\partial V(z)}{\partial z} \ell \left(\begin{array}{c}
z_2, \cdots, z_n, -K_\lambda z_{\lambda V(z)}\end{array}\right)^T + \sum_{i=1}^n \frac{\partial V(z)}{\partial z_i} (\phi_i(x_i) - \phi_i(x^*_i)) / \ell^{i-1}
$$

$$
\leq -\alpha \ell \hat{V}^{\frac{\tau}{2}}(z) + \sum_{i=1}^n \frac{\partial V(z)}{\partial z_i} (\phi_i(x_i) - \phi_i(x^*_i)) / \ell^{i-1}.
$$

(6)

To proceed, the following proposition, whose proof is laborious and included in the Appendix, is required.

Proposition 3.2: There exists a constant $\bar{\alpha} \in \mathbb{R}_+$ which is dependent on $\bar{\rho}$ and independent of $\ell$, such that the following relation holds

$$
\sum_{i=1}^n \left. \frac{\partial V(z)}{\partial z_i} (\phi_i(x_i) - \phi_i(x^*_i)) / \ell^{i-1} \right|_{\mathcal{U}_N} \leq \bar{\alpha} \hat{V}^{\frac{\tau}{2}}(z).
$$

Substituting the relation in Proposition 3.2 into (6) yields

$$
\dot{V}(z)_{\mathcal{U}} \leq -(\alpha \ell - \bar{\alpha}) \hat{V}^{\frac{\tau}{2}}(z).
$$

(7)

Now one can select a sufficiently large scaling gain $\ell \geq 1$ to satisfy the following guideline

$$
\alpha \ell - \bar{\alpha} \geq 1,
$$

(8)

which leads to

$$
\dot{V}(z)_{\mathcal{U}} \leq -\hat{V}^{\frac{\tau}{2}}(z).
$$

(9)

In what follows, we will use a contradiction argument to prove that under the guideline (8), for any non-zero initial states satisfying $x(0) \in \mathcal{U}_x$, all the trajectories of $z(t)$ will stay in $\Omega$ forever.

If the above statement is not true, the trajectory of $z(t)$ will escape the set $\Omega$ within a finite-time. Due to the fact that $z(0) \in \mathcal{U}_N$, it yields

$$
\dot{V}(z(0)) \leq -(\alpha \ell - \bar{\alpha}) \hat{V}^{\frac{\tau}{2}}(z(0)) \leq -\hat{V}^{\frac{\tau}{2}}(z(0)) < 0.
$$

(10)

Hence there must exist two time instants $t_1 > t_2 > 0$, such that

- $i$) $\dot{V}(z(t_1)) < 0$, $ii$) $V(z(t_2)) = V(z(t_1))$, $iii$) $\dot{V}(z(t_2)) > 0$.

(11)

It is clear that (9) still holds for $t \in [t_1, t_2]$. The following relations hold

$$
V(z(t_2)) - V(z(t_1)) = \int_{t_1}^{t_2} \dot{V}(z(s)) ds \leq -\int_{t_1}^{t_2} \hat{V}^{\frac{\tau}{2}}(z(s)) ds.
$$

(12)

By the relation ii) of (11) and the fact that $V(z(s)) > 0$, $s \in [t_1, t_2]$, (12) leads to an obvious contradiction, expressed as $0 \geq \int_{t_1}^{t_2} \hat{V}^{\frac{\tau}{2}}(z(s)) ds > 0$. Hence, we can arrive at the following conclusion $\forall x(0) \in \mathcal{U}_x \Rightarrow z(0) \in \mathcal{U}_x \Rightarrow z(t) \in \Omega$, $\forall t \geq 0$ which implies that the set $\Omega$ is an invariance set.

Further, by Lemma A.2 in mind and owing to the fact that $\tau \in (-\varepsilon, 0)$, hence $0 < \frac{\tau}{2} < 1$, the relation $\dot{V}(z)_{\mathcal{U}} \leq -\hat{V}^{\frac{\tau}{2}}(z)$ leads to a straightforward conclusion that there exists a finite time instant $T > 0$, such that $y(t) - y_r = 0$, $t \in [T, \infty)$. This completes the proof of Theorem 3.1.

Remark 3.1: In practices, different with local control strategies, the compact set $\mathcal{U}_x$ of the initial states can be defined
under an extreme case study. By following the guideline (8), one can determine a proper bandwidth factor \( \ell \) to meet the control performance specifications. On the other hand, the formula (8) could be somehow conservative due to the extensively used mathematical estimations. A practical “trial and error” way of the parameter configuration can be stated as follows. The control gain \( K \) can be simply selected following the pole placement manner first. By setting a large \( \ell \) to guarantee the stability, then one can tune \( \ell \) to be smaller and smaller while testing the gap between current control performance and pre-given performance indexes until satisfactory response curves appear.

IV. APPLICATION TO ROBUST MOTION CONTROL FOR SERIES ELASTIC ACTUATOR

A. System Description

Series Elastic Actuators (SEAs) are widely applied in advanced robot applications due to their advantages over conventional stiff and non-back derivable actuators in force control, e.g., high fidelity, low cost, low stiction, etc. [25]–[27]. As a sketch review of the related literature, linear controllers constitute a main choice, the performances can also be improved by using nonlinear control strategy and adding feed-forward control loops, see for examples [28]–[30], etc. However, it is worthy of pointing out that almost all those existing control applications to SEAs are based on an asymptotical control result. Due to the fact that both a fast convergence rate and a stronger robustness are highly demanded for SEA system in practices, finite-time regulation strategy will be of significance compared with existing related asymptotical control results. To the best of the authors’ knowledge, only one recent work [31] in the literature addresses the finite-time control problem for SEA by using a terminal sliding-mode control scheme, where the bothersome chattering issue cannot be avoided. In this section, we will show that the proposed theoretical result will provided a much easier control implementation, and meanwhile significant control performance improvements can be achieved compared with conventional PD and linear state feedback controllers, while there are not much added complexities of the gain tuning mechanism.

In this paper, we use a Series Elastic Actuator with a novel design (as depicted by Fig. 2) that gives the actuator different impedances at different force ranges. The actuator has two series elastic elements: a linear spring with a low stiffness and a torsional spring with a high stiffness. In this paper, we verify the proposed controller using only the torsional spring. Fig. 2 (a) is a cross section showing the structure of the studied actuator. The motor (Maxon EC-4pole brushless DC motor operating at 200W) shown is coupled to a ball screw through a torsional spring. Two incremental encoders (Renishaw RM22IC) with resolutions of 2048 and 1024 pulses per revolution are used to measure the angular displacement of the motor shaft and lead screw respectively. Using the analogy of two-mass-spring-damper system, by neglecting the inevitable unmodelled disturbances, one can obtain the nominal mathematical model of the following form [30]

\[
\begin{cases}
    m_m \ddot{q}_m + b_m q_m = F_m - k(q_m - q_l), \\
    m_l \ddot{q}_l + b_l q_l = k(q_m - q_l),
\end{cases}
\]

where the description of all involved parameters are listed in Table I.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_m )</td>
<td>inertia/mass of the motor</td>
</tr>
<tr>
<td>( m_l )</td>
<td>inertia/mass of the link</td>
</tr>
<tr>
<td>( b_m )</td>
<td>viscous friction coefficient of the motor</td>
</tr>
<tr>
<td>( b_l )</td>
<td>viscous friction coefficient of the link</td>
</tr>
<tr>
<td>( q_m )</td>
<td>angle/position of the motor</td>
</tr>
<tr>
<td>( q_l )</td>
<td>angle/position of the link</td>
</tr>
<tr>
<td>( k )</td>
<td>stiffness of the SEA</td>
</tr>
<tr>
<td>( F_m )</td>
<td>motor torque/force</td>
</tr>
</tbody>
</table>

Let \( x = [x_1, x_2, x_3, x_4]^T = [q_l, q_l, q_m, \dot{q}_m]^T \). System (13) can be expressed as the following state-space form

\[
\begin{align*}
    \dot{x}_1 &= x_2, \\
    \dot{x}_2 &= \frac{m_l}{m_m} x_3 - \frac{k}{m_m} x_1 - \frac{b_l}{m_m} x_2, \\
    \dot{x}_3 &= x_4, \\
    \dot{x}_4 &= \frac{1}{m_m} F_m - \frac{k}{m_m} (x_3 - x_1) - \frac{b_m}{m_m} x_4.
\end{align*}
\]

B. Finite-Time Robust Controller Design

In what follows, we consider a more practical control objective, namely, semi-global control, rather than the restrictive global control target. Hence, without any pre-verifications of certain nonlinearity growth conditions, it is straightforward to utilize the proposed tracking control approach to design a robust finite-time control law (the tracking reference is denoted by \( q_{\text{ref}} \)) to realize the accurate position control. With a series of pre-calculations as \( x_1^{(1)} = q_{\text{ref}}, \quad x_2^{(1)} = q_{\text{ref}}, \quad x_3^{(1)} = \frac{m_l}{m_m} q_{\text{ref}}, \quad x_4^{(1)} = \frac{m_l}{m_m} q_{\text{ref}}, \quad F_m = m_m \left( \frac{(x_3 - x_1)^2}{\ell} \right), \quad z_1 = x_1 - x_1^{*}, \quad z_2 = (x_2 - x_2^{*})/\ell, \quad z_3 = (x_3 - x_3^{*})/\ell, \quad z_4 = (x_4 - x_4^{*})/\ell \), then we are able to construct the following implementable finite-time control law

\[
\begin{align*}
    v &= -K \left[ z_1 \right]^{1+\ell} + \left[ z_2 \right]^{1+\ell} + \left[ z_3 \right]^{1+\ell} + \left[ z_4 \right]^{1+\ell}, \\
    F_m &= \ell^\ell v + F_m^*.
\end{align*}
\]
C. Experimental Setup

In the experimental setup, the control algorithm is implemented in real-time at 1KHz on a dSPACE DS1007 processor board with the DS3002 incremental encoder board for reading the encoders. The motor is controlled using the Elmo Gold Whistle Servo Drive which accepts motor torque commands as an analog signal. The dSPACE DS2102 DAC board is used to generate the motor command. The nominal values of the experimental SEA are identified as follows: \( m_m = 2.2 \times 10^{-6} \text{kg m}^2 \), \( m_l = 4 \times 10^{-8} \text{kg m}^2 \), \( k = 0.14 \text{N m rad}^{-1} \), and \( b_m = b_l = 1 \times 10^{-3} \text{N m s}^{-1} \text{rad}^{-1} \).

D. Parameter Configuration and Performance Verification

In what follows, starting from PD control, we will show how to establish the proposed practically oriented finite-time controller and elaborate the control performance improvement by choosing appropriate parameters of the proposed finite-time controller.

**Step 1: from PD control to state feedback control.** In the starting session, we first implement a conventional PD controller. Figs. 3-4 show the set-point and trajectory tracking performances under different proportional gains \( k_p = 0.3, 0.5, 0.7, 0.9 \) while \( k_d = 0.01 \), where \( q_{\text{ref}} = 0.5(\text{rad}) \) in Fig. 3 and \( q_{\text{ref}} = 0.5 \sin(5t + \varphi)(\text{rad}) \) in Fig. 4, respectively. Generally speaking, choosing a larger proportional gain will result in faster convergence rate, higher precision, but meanwhile, larger overshoot and control energy consumption. Without loss of generality, we extend the PD coefficients of the red dash lines in Figs. 3 and 4 as the coefficients of linear state feedback controller, i.e., \( K = [0.5, 0.01, 0.1, 0.01] \). Compared with the black dot line in Figs. 5 and 8, it is obvious to see the progressive tracking performances of state feedback controller.

**Step 2: from state feedback control to finite-time control.** Firstly notice that the proposed finite-time control law (15) reduces to a linear state feedback controller if we set the homogeneous degree \( \tau = 0 \). In this step, by simply modifying the homogeneous degree \( \tau \) from 0 to several negative values gradually, we can implement the proposed finite-time controller to obtain a better control performance while the control gains can be set as fix values. By understanding that in real-life systems, there are various disturbances/uncertainties, hence it is of significance that the proposed finite-time controller could reduce the settling time and improve the system robustness against the inevitable disturbances/uncertainties. As depicted by Figs. 5 and 8, the control performance is significantly improved if \( \tau \) is a negative value and moreover, a smaller \( \tau \) will clearly lead to a faster convergence speed and lower steady error. Under the proposed finite-time controller (15), it can be observed from Figs. 6 and 9 that the bandwidth factor \( \ell \) has also played a key role as a larger \( \ell \) will lead to an obvious performance variation as well. However, it should be pointed out here that a larger \( \ell \) will cause a clear deterioration of the system robustness against the measurement noises, which is a common problem of existing high gain control methods. In Figs. 7 and 10, the control performance variation along with
the control gain selection $k_l$ while the other control parameters are fixed is depicted. To make the comparisons clearer and more precise, the performance indexes (overshoot, offset) of set-point tracking and ISE index ($\int_0^T e^2(t)dt$) where $[t_1, t_2]$ is a period of the reference signal in the steady state and $e(t)$ is the tracking error) for trajectory tracking case are included in Table II.

As a direct conclusion from the above illustrated figures, the proposed finite-time control strategy will clearly lead to a significant control performance improvement while the control gain selection guideline is as simple as conventional linear state feedback controllers. Moreover, the added negative homogeneous degree will endow the control engineers a much flexibility of tuning the control performances in practical implementations.

Fig. 5: Set-point tracking performances under finite-time controller (15) with different homogeneous degree $\tau$ while $K = [0.5, 0.01, 0.1, 0.01]$ and $\ell = 1$.

Table II. Performance Indexes of PD controller, linear state feedback controller and finite-time controller.

<table>
<thead>
<tr>
<th>Methods</th>
<th>Parameters</th>
<th>Overshoot</th>
<th>Offset</th>
<th>ISE</th>
</tr>
</thead>
<tbody>
<tr>
<td>PD</td>
<td>$k_p = 0.3$</td>
<td>0.00</td>
<td>-20.21</td>
<td>751.14</td>
</tr>
<tr>
<td></td>
<td>$k_p = 0.5$</td>
<td>8.68</td>
<td>19.87</td>
<td>425.32</td>
</tr>
<tr>
<td></td>
<td>$k_p = 0.7$</td>
<td>4.90</td>
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E. Performance Comparison with Optimal Controller

In order to better demonstrate the control performance superiority of the proposed nonsmooth controller with existing asymptotical controllers, we present a experimental performance comparison with an optimal controller by predictive approach [32]–[34]. The controller is derived based on optimizing a performance index $J(t) = \frac{1}{2} \int_0^T (x_1(t + \tau) - x_1^*(t + \tau))^2 d\tau$, where $T$ is the predictive period. Utilizing a predictive approach associated with the Taylor expansion, the optimal controller is derived in the form of

$$F_m^{Opt} = -\sum_{i=1}^{4} k_i^{Opt} (x_i - x_i^*).$$

The detail process of derivation can be found in [32]. It is worth noting that the optimal gains $k_i^{Opt}, i \in \mathbb{N}_{1:4}$ are only related to the predictive period $T$ and the control order $r$. For simplicity, the control order $r$ is set as 0 and the predictive period $T$ is set as the only tunable parameter.
Fig. 8: Trajectory tracking performances under finite-time controller (15) with different homogeneous degree \( \tau \) while \( K = [0.5, 0.01, 0.1, 0.01] \) and \( \ell = 1 \).

Under a set-point tracking control objective, the control performance comparisons of the proposed finite-time controller (15) and the optimal controller (16) are presented in Fig. 11. By noting that the initial torque amplitude of the candidate controllers are placed in a similar level in order to make a fair comparison, the robustness exhibited by the proposed finite-time controllers are much stronger than the optimal controllers. A lower steady state error can be achieved with the import of a negative homogeneous degree. Similar conclusions can also be obtained from the case of trajectory tracking as shown in Fig. 12. To make the comparison more clearer, the detailed performance indexes of both set-point tracking and trajectory tracking cases are also included in Table III.

V. Conclusions

In this paper, we investigate a novel non-recursive tracking control design framework under a semi-global control objective for a class of lower-triangular nonlinear systems. Compared with all existing related results, several improvements are achieved. Firstly, the proposed control scheme is much simpler and the control gain can be easily selected following a pole placement pattern. Secondly, it is shown that a finite-time trajectory tracking result can be realized for smooth nonlinear systems without any additional nonlinearity growth conditions. Moreover, the proposed one-step control design and stability analysis under a new non-recursive synthesis manner will facilitate the parameter figuration and practical implementations. An application to a SEA system and experimental performance comparison results are provided to illustrate the simplicity and effectiveness of the proposed controller. The proposed

Table III. Performance Indexes of optimal controller and finite-time controller.

<table>
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<th>Offset</th>
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<td>0.18</td>
<td>-0.52</td>
<td>23.54</td>
</tr>
<tr>
<td></td>
<td>FTC2 44.97</td>
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<td>-0.52</td>
<td>10.45</td>
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</table>
control strategy could provide a more efficient implementation choice for those real-life systems with high demand of fast control speed and stronger robustness. Future works will focus on the finite-time active disturbance/uncertainty attenuation problem for systems with large modeling error and external disturbances.

**Appendix**

### A. Useful Lemmas

Some useful lemmas are stated as follows for the convenience of readers.

**Lemma A.1:** [6] Let $V_1(x) \in \mathbb{H}^1_N$ and $V_2(x) \in \mathbb{H}^2_N$, respectively, then the following statements hold.

i) $V_1(x)V_2(x) \in \mathbb{H}^{1+2}_N$, ii) $\frac{\partial V_1(x)}{\partial x_1} \in \mathbb{H}^{-1+1}_N$, $i \in \mathbb{N}_{1,N}$. iii) If $V_1(x)$ is positive definite, then the following relation holds $(\min_{|V_1(x)|=1} V_2(x)) V_1^{1/2} (x) \leq V_2(x) \leq (\max_{|V_1(x)|=1} V_2(x)) V_1^{1/2} (x)$.

**Lemma A.2:** [6] Consider a dynamical system $\dot{x} = f(x,t)$, $f(0,t) = 0$. Suppose there exists a $C^1$ positive-definite and proper function $V: \mathbb{R}^n \rightarrow \mathbb{R}^n$ and real numbers $c > 0$ and $r \in (0, 1)$, such that $V + cV^r$ is semi-negative definite, then the origin $x = 0$ is a globally finite-time stable equilibrium with a settling time $T \leq \frac{V(0)}{c(1-r)}$ for any given initial condition $x_0 = x(t_0)$.

**Lemma A.3:** [6], [23] Consider the following chain of integrators

$$\dot{\eta}_i(t) = \eta_{i+1}(t), \ i \in \mathbb{N}_{1,n-1}, \ \eta_n(t) = u(t),$$

under a homogeneous control law of the following form

$$u(t) = -K\eta_1^{1+r}$$

where $K$ is the coefficient vector of a Hurwitz polynomial $s^n + k_n s^{n-1} + \cdots + k_2 s + k_1 > 0$, $\tau > -\frac{1}{n}$ is a homogeneous degree. There exists a constant $\epsilon \in (0, \frac{1}{n})$, such that the closed-loop system (17)-(18) is globally finite-time stable for $\tau \in (-\epsilon, 0)$.

### B. Proofs of Propositions

This subsection collects the proofs of propositions used in the paper.

**Proof of Proposition 3.1:** By using Lemma A.3, one can obtain the conclusion that $\frac{\partial V(z)}{\partial z} (z_2, \cdots, z_n, -K[z]^{1+r}_N)^T$ is negative definite for a homogeneous degree $\tau \in (-\epsilon, 0)$. With $V(z) \in \mathbb{H}^{1+r}_N$ and $(z_2, \cdots, z_n, -K[z]^{1+r}_N)^T \in \mathbb{H}^{1+r}_N$ in mind, using Lemma A.1, the following relation can be achieved for a constant $\epsilon \in \mathbb{R}_+$

$$\frac{\partial V(z)}{\partial z} \ell (z_2, \cdots, z_n, -K[z]^{1+r}_N)^T \leq -\alpha t \|z\|^2_N.$$  

**Proof of Proposition 3.2:** Recalling $f_i$, $i \in \mathbb{N}_{1,n}$ is a smooth function, by utilizing Mean-Value Theorem, we have

$$\phi_i(x_i) - \phi_i(x_i^*) \leq \frac{\partial \phi_i}{\partial \xi_i} (x_i - x_i^*) \leq |x_i - x_i^*|.$$
In the case when 
We know that 
It is clear that 
\[ V(z) \in \mathbb{H}_N^{-r} \] and 
\[ \|x\|_{-r}^1 \in \mathbb{H}_{N}^{-r}. \] With Lemma A.1 in mind, there exists a constant \( \alpha \in \mathbb{R}^+ \) which is dependent on \( N \) but independent of \( \ell \), such that the following relations hold
\[
\sum_{i=1}^{n} \frac{\partial V(z)}{\partial z_i} (x_i - \phi_i(x_i)) / \ell^{i-1} \leq \tilde{y}_i \|x\|_{-r}^1, \tag{21}
\]
\[
\sum_{i=1}^{n} \frac{\partial V(z)}{\partial z_i} \|x\|_{-r}^1 \leq \tilde{\alpha} \tilde{V}^{1/2}(z). \tag{22}
\]

**References**


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