Brief Paper

Fast and low-frequency adaptation in neural network control

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Abstract: In adaptive neural network (NN) control, fast adaptation through high-gain learning rates can cause high-frequency oscillations in control response resulting in system instability. This study presents a simple adaptive NN with proportional derivative (PD) control strategy to achieve fast and low-frequency adaptation for a class of uncertain non-linear systems. Variable-gain PD control without the knowledge of plant bounds is proposed to semi-globally stabilise the plant, so that NN approximation is applicable. A low-pass filter-based modification is applied to the adaptive law to filter out high-frequency content, so that tracking performance can be safely improved by the increase of learning rates. The novelties of this study with respect to adaptive NN control are as follows: (i) semi-global practical asymptotic tracking can be achieved by a simple adjustment of control parameters; and (ii) fast and low-frequency adaptation can be obtained via high-gain learning rates under guaranteed system stability. Simulation studies have demonstrated that the proposed approach can outperform its non-filtering counterpart in terms of tracking accuracy, energy cost and control smoothness.

1 Introduction

Adaptive neural control (ANC) is a type of adaptive control strategies for tackling non-parametric uncertainties in non-linear systems [1] and has attracted great concern in recent years. For example, see [2–20] and some references therein. There are at least three time scales in adaptive control systems [21], namely a time scale of the underlying closed-loop dynamics with fixed plant parameters and controller parameters, a time scale associated with plant identification, and a time scale of plant parameter variations [22]. Typically, for a stable and safe adaptive control system, the identification time scale should be faster than the other two time scales. However, increasing the identification time scale to achieve fast adaptation by means of high-gain learning rates can cause high-frequency oscillations in control response, which can excite unmodelled system dynamics resulting in system instability [23]. Thus, there is a critical trade-off between system stability and adaptation gain.

To deal with this trade-off, model reference adaptive control with guaranteed fast and low-frequency adaptation was developed for a class of affine non-linear systems with parametric uncertainties in [24], where a low-pass filter is introduced in the adaptive loop to guarantee fast and safe adaptation under high-gain learning rates. There are mainly two issues for the extension of the approach in [24] to the ANC of non-linear systems with non-parametric uncertainties. One is to guarantee global or semi-global closed-loop stability, so that the neural network (NN) approximation is applicable. The other is to ensure transient and steady-state tracking performances of the resulting closed-loop system. Several approaches, including discontinuous control compensation [25], adaptive bounding [1], global NN approximation [26] and proportional derivative (PD) control [27], have been applied to globally or semi-globally stabilise ANC systems. Yet, discontinuous control is inapplicable in many practical problems, and adaptive bounding and global NN approximation inevitably increase implementation cost. Since most ANC structures include PD control terms, applying PD control is relatively simple but effective since it does not complicate the control structure.

Motivated by the aforementioned two issues, this paper presents a simple ANC with PD control strategy to guarantee fast and low-frequency adaptation for a class of uncertain non-linear systems. The procedure of control design is as follows: firstly, a linearly parameterised NN is applied to approximate a generalised uncertainty; secondly, variable-gain PD control without the knowledge of plant bounds is proposed to obtain semi-global closed-loop stability; thirdly, a low-pass filter-based modification is applied to the adaptive law to filter out high-frequency content; finally, the transient and steady-state tracking results are obtained by Lyapunov-based analysis. It is shown that a domain of global stabilisation can be covered by a domain of regional attraction via a simple adjustment of control parameters, which guarantees semi-global practical asymptotic tracking of the closed-loop system. Although this study is an extension of [24], because of the introduction of NN approximation, extra

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efforts should be made to establish the semi-global practical asymptotic stability result.

The rest of this paper is organised as follows. The considered problem is formulated in Section 2. The ANC with PD control strategy is proposed in Section 3. An illustrative example is demonstrated in Section 4. Finally, conclusions are given in Section 5. Throughout this paper, \( \mathbb{R} \), \( \mathbb{R}^+ \) and \( \mathbb{R}^n \) denote the spaces of real numbers, positive real numbers and real \( n \)-vectors, respectively. \(|\cdot|\) and \( \|\cdot\|\) denote the absolute value and the standard Euclidean norm, respectively, \( L_2 \) and \( L_\infty \) denote the spaces of square integrable signals and essentially bounded signals, respectively, diag(\( \cdot \)) denotes the function of diagonal matrix, \( \min(\cdot) \), \( \max(\cdot) \) and \( \sup(\cdot) \) represent the functions of minimum, maximum and supremum, respectively, and \( C^k \) represents the space of functions whose \( k \)-order derivatives all exist and are continuous, where \( n \) and \( k \) are non-negative integers.

## 2 Problem formulation

Consider the following class of affine non-linear systems in the Brunovsky canonical form [28]

\[
\begin{align*}
\dot{x}_i &= x_{i+1} \quad (i = 1, 2, \ldots, n - 1) \\
\dot{x}_n &= f(x) + g(x_{n-1})u \\
y &= x_1
\end{align*}
\]

(1)

where \( x(t) := [x_{n-1}(t)]^T, x_n(t) \in \mathbb{R}^n \) with \( x_{n-1}(t) := [x_1(t), x_2(t), \ldots, x_{n-1}(t)]^T \) is the measurable state vector, \( u(t) \in \mathbb{R} \) and \( y(t) \in \mathbb{R} \) are the control input and system output, respectively, \( f(x) : \mathbb{R}^n \mapsto \mathbb{R} \) satisfying \( f(0) = 0 \) is the unknown \( C^1 \) non-linear driving function, and \( g(x^{u-1}) : \mathbb{R}^{n-1} \mapsto \mathbb{R} \) is the unknown \( C^1 \) control gain function.

**Assumption 1** [28]: There exist an unknown finite constant \( g_0 \in \mathbb{R}^+ \) and an unknown continuous function \( \tilde{g}(x_{n-1}) \), such that \( 0 < g_0 \leq \tilde{g}(x_{n-1}) \leq \tilde{g}(x_{n-1}), \forall x_{n-1} \in \mathbb{R}^{n-1} \).

**Assumption 2** [28]: The desired output \( y_d(t) \) satisfies \( y_d(t) = C x(t) + d(t) \), where \( C \) is a finite constant determined by activation functions, and \( d(t) \in \mathbb{R}^m \) is a finite constant. Then, one obtains

\[
\dot{e}_i = f(x) + g(x_{n-1})u + v
\]

(2)

with \( v := -y^{(m)} + [0, \Lambda^2]^T e \). It follows from the Lemma 4.2 of [28] that the following result holds.

**Lemma 1**: For the system in (1) satisfying Assumptions 1 and 2, an ideal control law is selected as follows

\[
u^* = -k e_i - (f(x) + v)/g(x_{n-1}) + h(x_1)
\]

(3)

where \( x_1 \in [e_1, X_1]^T \in \mathbb{R}^{n+2}, k \in \mathbb{R}^+ \) is a control gain parameter, and \( h(\cdot) \) is given by

\[
h(x_1) := \tilde{g}(x_{n-1}) e_i/(2g^2(x_{n-1}))
\]

(4)

Then, one has \( \lim_{t \to \infty} ||e(t)|| = 0 \).

Yet, the ideal control law in (2) is inapplicable since \( f(\cdot) \) and \( g(\cdot) \) are unknown a priori. The control objective of this study is to determine an ANC plus PD control strategy for the system in (1) under Assumptions 1 and 2 such that \( y \) semi-globally tracks \( y_d \) as fast and accurate as possible.

**Remark 1**: For the simplification of discussion, the single-input–single-output (SISO) affine non-linear system in (1) with \( g(\cdot) > 0 \) is considered in this study. From Remark 1 of [27], the following results can be easily modified to the system in (1) with \( g(\cdot) < 0 \). In addition, a class of SISO non-affine non-linear systems can be transformed into the form of (1) by the approach of [29], and a class of multi-input–multi-output (MIMO) non-affine non-linear systems can be transformed into several SISO non-affine non-linear systems as shown in [30]. Thus, the results obtained based on (1) are possible to be extended to the MIMO non-affine non-linear system under certain conditions.

## 3 Fast and low-frequency adaption

### 3.1 Closed-loop system dynamics

For clear presentation, define a generalised uncertainty

\[
F(x_1) := (f(x) + v)/g(x_{n-1}) - h(x_1)
\]

(5)

which is of class \( C^1 \) since all \( f(\cdot) \), \( g(\cdot) \) and \( h(\cdot) \) are of class \( C^1 \). To approximate \( F(\cdot) \) in (5), one introduces a class \( C^1 \) linearly parameterised NN [1]

\[
\hat{F}(x_1, \hat{W}) = \hat{W}^T \Phi(x_1) = \sum_{j=1}^M \hat{w}_j \phi_j(x_1)
\]

(6)

where \( \Phi(x) := [\phi_j(\cdot), \phi_2(\cdot), \ldots, \phi_M(\cdot)]^T : \mathbb{R}^{n+2} \to \mathbb{R}^M \) satisfying \( ||\phi(\cdot)|| \leq \psi \) is the vector of basis functions, \( \hat{W} := [\hat{w}_1, \hat{w}_2, \ldots, \hat{w}_M]^T \in \mathbb{R}^M \) is the vector of weights, \( \psi \in \mathbb{R}^+ \) is a finite constant determined by activation functions, and \( M \) is the number of neurons. Then, the certainty equivalent control law is designed as follows

\[
u(x_1, \hat{W}) = -k e_i - \hat{F}(x_1, \hat{W})
\]

(7)

where \( u_1 \), essentially, is a PD control term. Define compact sets \( \Omega_{x_1} := \{x_1 | x \in D_1, y_d \in \Omega_d\}, D_1 := \{y_1 \leq C_1, \mu_d := ||y_d|| \leq C_d, \mu_d := ||y_d|| \leq C_d\} \) and \( \Omega_{x_1} := [\hat{W}^T \|\hat{W}\| \leq C_d] \), where \( D_1 \) is a domain of NN approximation, and \( C_1, C_d, \mu_d \in \mathbb{R}^+ \) are prespecified finite constants. Next, define an optimal NN approximation error \( \varepsilon \) as follows

\[
\varepsilon(x_1) := F(x_1) - \hat{F}(x_1, W^*)
\]

(8)

where \( W^* \) is a vector of optimal weights given by

\[
W^* := \arg \min_{W \in D_2} \left( \sup_{x_1 \in \Omega_{x_1}} ||F(x_1) - \hat{F}(x_1, \hat{W})|| \right)
\]

Adding and subtracting \( g(\cdot)h(\cdot) \) on the right side of (2) and making some transformations leads to

\[
\dot{e}_i = g(x_{n-1}) (u + F(x_1) + h(x_1))
\]

(9)
Using (7)–(9) and making some manipulations, one obtains the closed-loop system dynamics as follows

$$\dot{e}_i = g(x_{i-1}) \left( \hat{W}^T \Phi(x_i) + e_i(x_i) + h(x_i) - k e_i \right)$$

(10)

in which $\hat{W} := W^* - \hat{W}$. From the universal approximation property of NNs in [1], one has the following lemma.

**Lemma 2 [1]:** The optimal approximation error $\varepsilon$ in (8) can be bounded by a constant $\mathcal{E} := \sup_{x_i \in \mathcal{S}_w} |e(x_i)|$ dominated by the number of neurons.

### 3.2 Semi-global Robust stabilisation

Let $\mathcal{D}_g := \{x||x|| \leq c_i \subset D_i$, with $c_i < c$, be a domain of global stabilisation, and $\mathcal{S}_g \subset \mathbb{R}^n$ be a domain of global attraction. To make the NN approximation in Section 3.1 applicable, one should ensure $x \in \mathcal{D}_g$, such that $x_i \in \mathcal{S}_w$. Choose a Lyapunov function candidate for (2) as follows

$$V_i(e_i) = e_i^2 / 2$$

Then, we give the first main result of this study.

**Theorem 1:** For the system in (1) satisfying Assumptions 1 and 2, the control law is designed as $u = -k(t)e_i$ with

$$k(t) = (1 + ||x(t)|| + |v(t)|) / \mu$$

(12)

where $\mu \in \mathbb{R}^+$ is a control parameter. Then there exist suitably large $\lambda$ and $1/\mu$ such that the closed-loop system achieves semi-global uniformly ultimately bounded (UUB) stability in the sense that $x(t) \in \mathcal{D}_g$ given by (13) after a finite time $T_i \geq 0$, $\forall x(t) \in \mathcal{S}_g$, where $\mathcal{S}_g$ can be arbitrarily enlarged by the increase of $\lambda$ and $1/\mu$.

**Proof:** From the expression of $V_i$ in (11), there exist class $\mathcal{K}_\infty$ functions $U_{i1}(e_i)$ and $U_{i2}(e_i)$, such that

$$U_{i1}(e_i) \leq V_i(e_i) \leq U_{i2}(e_i)$$

$\forall t \geq 0$ and $e_i \in \mathbb{R}$, where $U_{i1}(e_i) := |e_i|^{2} / 2$ and $U_{i2}(e_i) := |e_i|^{2}$. As $f(\cdot)$ is of class $\mathcal{C}^1$ and $f(0) = 0$, it can be obtained from the Mean Value Theorem that $f(\cdot) \leq \lambda ||x||$, $\forall x \in \mathcal{S}_g$, where $\lambda \in \mathbb{R}^+$ is a finite constant. Differentiating $V_i$ along (2) with respect to time $t$ and applying $u = -k(t)e_i$, $f(\cdot) \leq \lambda ||x||$ and Assumption 1, one obtains

$$\dot{V}_i = e_i(f(x) - g(x_i))k(t)e_i + v$$

$$= -k(t)g(e_i^2 - e_i(f(x) + v)/(k(t)g))$$

$$\leq -k(t)g(e_i^2 - |e_i|\lambda ||x|| + |v|)/(k(t)g_0)$$

Letting $\delta := \max\{1/g_0, \lambda/g_0\}$, one obtains

$$\dot{V_i} \leq k(t)g(e_i^2 - \delta |e_i| ||x|| + |v|) / k(t)$$

Substituting (12) to the above expression leads to

$$\dot{V}_i \leq k(t)g(|e_i|^2 - \delta |e_i|)$$

Thus, if $|e_i| \geq \delta \mu$, then one obtains

$$\dot{V_i} \leq -k(t)g(|e_i|^2 - \delta \mu|e_i| - \delta \mu(|e_i| - \delta \mu))$$

$$= -k(t)g(|e_i|^2 - \delta \mu)^2$$

From Assumption 1 and (12), one has $k(t)g \geq g_0 / \mu$. Applying this result to the above expression yields

$$\dot{V}_i \leq -U_{i2}(e_i), \forall |e_i| \geq \delta \mu$$

in which $U_{i2}(e_i) := g_0(|e_i| - \delta \mu)^2 / \mu \in \mathbb{R}^+$. Accordingly, from the Theorem 4.18 of [31], there exist a class $\mathcal{KL}$ function $U_{i2}(e_i(0), t)$ and a finite time $T_i = T_i(e_i(0), \delta \mu)$ such that the solution of (2) satisfies

$$|e_i(t)| \leq U_{i2}(e_i(0), t), \forall 0 \leq t \leq T_i$$

$$|e_i(t)| \leq U_{i2}^{-1}(U_{i2}(\delta \mu)), \forall t \geq T_i$$

which implies $e_i$ is UUB by

$$b_i := U_{i2}^{-1}(U_{i2}(\delta \mu)) = \sqrt{2\delta \mu}, \forall t \geq T_i$$

From the properties of the filtered tracking error $e_i$ (see Remark 2.1 of [27]), one obtains $|e_i| \leq 2\lambda^{-1}\sqrt{2\delta \mu}$ with $i = 1, 2, \ldots, n$, $\forall \delta \geq T_i := T_i + (n - 1)^{1/\lambda}$. Thus, $x$ exponentially converges to the domain $\mathcal{D}_g$ given by

$$\mathcal{D}_g := \{x||x|| \leq 2\lambda^{-1}\sqrt{2\delta \mu},$$

$$i = 1, 2, \ldots, n, y_d \in \mathcal{S}_g\}, \forall t \geq T_i$$

which is a positively invariant set of $x$. It is shown from (13) that $\mathcal{D}_g$ can be contracted by the decrease of $\delta$ (as well as $\lambda$, $1/\lambda$ and $\mu$). Combining with the fact that the size of $\mathcal{S}_g$ is positively correlated with $\lambda$, one that $\mathcal{D}_g \subset \mathcal{S}_g$ should be satisfied to guarantee closed-loop stability, $\forall x(0) \in \mathcal{S}_g$. Since the size of $\mathcal{D}_g$ is negatively correlated with $\lambda$, $1/\lambda$ and $\mu$, one concludes that: (i) There must exist suitably large $\lambda$ and $1/\mu$ such that $\mathcal{D}_g \subset \mathcal{S}_g$ holds for a given $\mathcal{S}_g$ related to $\lambda$, that is, the closed-loop system achieves UUB stability [32]; and (ii) The stability result can be maintained for an arbitrarily large $\mathcal{S}_g$ (as well as $\lambda$) by the increase of $\lambda$ and $1/\mu$, that is, the stability result is semi-global [32].

**Remark 2:** Theorem 1 implies that $x(t)$ can exponentially converge to $\mathcal{D}_g$ after a finite time $T_i \geq 0$, $\forall x(0) \in \mathcal{S}_g$ under the proposed control law, where $\mathcal{S}_g$ can be arbitrarily enlarged by the increase of $\lambda$ and $1/\mu$. It is observed that the size of $\mathcal{D}_g$ in (13) is unclear since $\delta$ is not easy to be exactly determined. However, since $x(t)$ is measurable, it is easy to make $x(t)$ be inside a desired $\mathcal{D}_g$ by the adjustment of $\lambda$ and $\mu$ in experiments. Semi-global stabilisation of ANC systems is useful, while $x(0)$ is initially outside of $\mathcal{D}_g$, or $x(t)$ becomes outside of $\mathcal{D}_g$ under sudden changes of plant dynamics or large external disturbances.

### 3.3 Regional tracking control

The results in Section 3.2 guarantee that $x(t) \in \mathcal{D}_g \subset D_i$ and $x_i(t) \in \mathcal{S}_w$, $\forall t \geq T_i$, so that the NN approximation in Section 3.1 is applicable. For facilitating denotation, reset $i = 0$. Then, consider the regional tracking problem $x(0) \in \mathcal{D}_g$. Normally, a learning rate cannot be made arbitrarily large since it can result in high-frequency oscillations.
at control response [23]. To tackle this issue, \( \hat{W}_f \), a low-pass filtered version of \( \hat{W} \), is introduced as follows

\[
\hat{W}_f = \alpha_f (\hat{W} - \hat{W}_f), \quad \hat{W}_f(0) = \hat{W}(0)
\]

where \( \alpha_f \in \mathbb{R}^+ \) is a prespecified filter time constant which should be suitably small to cut off the high-frequency content of \( \hat{W} \). The adaptive law of \( \hat{W} \) is designed as follows

\[
\dot{\hat{W}} = \text{proj}(\gamma e, \Phi(x_u) - \gamma \sigma(\hat{W} - \hat{W}_f))
\]

where \( \gamma \in \mathbb{R}^+ \) is a learning rate, \( \sigma \in \mathbb{R}^+ \) is a small constant, and \( \text{proj}(\bullet) \) is a projection operator given by [1]

\[
\text{proj}(\bullet) = \begin{cases} 
\bullet & \text{if } \|\hat{W}\| < c_w \\
\text{or } \|\hat{W}\| = c_w \text{ and } \hat{W}^T \bullet \leq 0 \\
\bullet - \hat{W} \hat{W}^T \bullet / \|\hat{W}\|^2 & \text{or } \|\hat{W}\| = c_w \text{ and } \hat{W}^T \bullet > 0
\end{cases}
\]

Choose the following Lyapunov function candidate

\[
V(z) = e^T/(2g) + \hat{W}^T \hat{W} / (2\gamma) + \sigma \hat{W}^T \hat{W}_f / (2\alpha_f)
\]

for the closed-loop system composed of (10), (14) and (15), where \( z := [e, \hat{W}, \hat{W}_f]^T \) and \( \hat{W}_f := \hat{W} - \hat{W}_f \). For the convenience of presentation, define two constants

\[
\begin{cases} 
\bar{c}_w := \max_{\|\hat{W}\| < c_w} \{\hat{W}^T \hat{W} / (2\gamma)\} \\
\bar{c}_f := \max_{\|\hat{W}\| < c_w} \{\sigma \hat{W}^T \hat{W}_f / (2\alpha_f)\}
\end{cases}
\]

Now, we establish the second main result of this study.

**Theorem 2:** Consider the system in (1) under Assumptions 1 and 2 driven by the control law in (7) with (12), (14)–(16). If \( x(0) \in D_x \) in (13) and \( \hat{W}(0) \in \Omega_x \), then there exist suitably large \( \lambda, 1/\mu \) in (12) and \( \gamma \) in (15) such that the closed-loop system achieves practical asymptotic stability with the following control performances: (i) The boundedness of all closed-loop signals; (ii) the transient bound of \( e_t \) in (21); and (iii) the steady state of \( e \) in (24).

**Proof:** (i) **The boundedness of all involved closed-loop signals.** Differentiating (17) along (10) with respect to time \( t \) and using (4), one obtains

\[
\dot{V} = e_t \dot{e} / g - e_t^2/(2g^2) - \hat{W}^T \hat{W} / \gamma - \sigma \hat{W}^T \hat{W}_f / \alpha_f
\]

\[
= -k(t)e_t^2 + e_t e(x_u) - \sigma \hat{W}^T \hat{W}_f / \alpha_f
\]

\[
+ \hat{W}^T(e_t \Phi(x_u) - \hat{W} / \gamma)
\]

From the results of the projection operator in [1], the adaptive law in (15) guarantees that: (i) \( \dot{V}(t) \in \Omega_x, \forall t \geq 0 \) if \( \hat{W}(0) \in \Omega_x \); and (ii) \( \hat{W}^T(e_t \Phi(x_u) - \hat{W} / \gamma) \leq \sigma(\hat{W} - \hat{W}_f) \). Applying these results with (14) and (15) to the above expression, one obtains

\[
\dot{V} \leq -k(t)e_t^2 + e_t e(x_u) + \sigma(\hat{W} - \hat{W}_f)^T(\hat{W} - \hat{W}_f)
\]

Using the fact \( \hat{W} - \hat{W}_f = \hat{W}_f - \hat{W} \), one obtains

\[
\dot{V} \leq -k(t)e_t^2 + e_t e(x_u) - \sigma \|\hat{W} - \hat{W}_f\|^2 \leq -k(t)e_t^2 + e_t e(x_u)
\]

where \( \sigma = c_w + \bar{c}_f + (\bar{c}_w / \mu)^2 / (2g_0) \in \mathbb{R}^+ \) is a finite constant.

(ii) **Transient tracking performance with a domain of attraction.** Combining (20) with (17), one obtains

\[
V(t) \leq e^{-g_0t/\mu} V(0) + \varphi, \quad \forall t \geq 0
\]

which implies \( V(t) \in L_\infty \). Combining with (14) and (17), \( y_d \in L_\infty \) and the definitions of \( c_w \) and \( \bar{c}_f \) in (21), one obtains \( e_t \), \( \hat{W}_f \), \( \hat{W}_f \), \( x \), \( v \) \in \( L_\infty \). Since \( e_t \), \( \hat{W}_f \), \( \hat{W}_f \), \( x \), \( v \) \in \( L_\infty \), and \( \|\Phi(\cdot)\| \leq (\gamma \sigma, (\gamma + 1)\sigma, \gamma \sigma, (\gamma + 1)\sigma) \) in the corresponding definition in (32).

(iii) **Transient tracking performance with a domain of attraction.** Combining (20) with (17), one obtains

\[
e_t^2 / (2g_1) \leq e^{-g_0t/\mu} V(0) + \varphi, \quad \forall t \geq 0
\]

in which \( g_1 := \max_{x \in \Omega_x} g_1(x) \in \mathbb{R}^+ \) is a finite constant. Thus, \( e_t \) can be bounded by

\[
|e_t| \leq e^{-g_0t/\mu} \sqrt{2g_1 V(0) + 2g_1 \varphi}, \quad \forall t \geq 0
\]

From (20) and the definition of \( \varphi \), increasing \( t \) and/or \( 1/\mu \) decreases \( \gamma \). Accordingly, since \( \alpha_f \), \( \gamma \), \( g_1 \) in (21) are positive constants and \( e_t \) depends on \( x(0) \), there exists a constant \( C_\gamma = C_\gamma(\mu, \gamma, x(0), \hat{W}(0)) \in \mathbb{R}^+ \), such that

\[
|e_t| \leq C_\gamma(\mu, \gamma, x(0), \hat{W}(0)), \quad \forall t \geq 0
\]

where \( C_\gamma \) can be decreased by the increase of \( 1/\mu \) and/or \( \gamma \). Define a maximal domain of attraction \( \Omega_x \) as follows

\[
\Omega_x := \{x(0)||x|e_t| \leq C_\gamma(\mu, \gamma, x(0), \hat{W}(0)) \}
\]

which implies that for all \( x(0) \in \Omega_x \) and \( y_d \in \Omega_d \), it has \( x(0) \in \Omega_x, \forall t \geq 0 \). Then, for the closed-loop system under a certain \( x(0) \in \Omega_x \) with \( \hat{W}(0) \in \Omega_x \) and \( y_d \in \Omega_d \), one can determine two constants \( \gamma^* \) and \( \mu^* \) as follows

\[
\gamma^* := \inf_{\gamma \in \mathbb{R}^+} \left\{ \gamma |x|e_t| \leq C_\gamma(\mu, \gamma, x(0), \hat{W}(0)) \right\}
\]

\[
\subset \Omega_x, y_d \in \Omega_d
\]

\[
\mu^* := \sup_{\mu \in \mathbb{R}^+} \left\{ \mu |x|e_t| \leq C_\gamma(\mu, \gamma, x(0), \hat{W}(0)) \right\}
\]

\[
\gamma^* \leq \gamma \subset \Omega_x, y_d \in \Omega_d
\]

Thus, for a certain \( x(0) \in \Omega_x \) with \( \hat{W}(0) \in \Omega_x \) and \( y_d \in \Omega_d \), if \( 1/\mu \) in (12) and \( \gamma \) in (15) are chosen to be suitably
large, such that $\gamma \geq \gamma^*$ and $\mu \leq \mu^*$, then $x(i) \in \Omega_x, \forall t \geq 0$. Therefore for the control law in (7) with (12), (14) and (15) satisfying $\gamma \geq \gamma^*$ and $\mu \leq \mu^*$, one can determine a corresponding domain of attraction

$$ S := \{x(0)||x(0)|| \leq C_{r}^{-1}(\mu, \gamma, e_T, \vec{W}(0), e_i, \bar{\xi}) ; e_i \in \Omega_i \} \quad (23) $$

where $\Omega_i := \{e_i|x \in \Omega_x, y_d \in \Omega_d\}$, and $C_r^{-1}(\cdot)$ is the inverse function of $C_r(\cdot)$ with respect to $x(0)$. According to (23) and $\mathcal{D}_{x} \subset \mathcal{D}$, the domain $S$ can be enlarged to satisfy $S \supseteq \bar{S}_{x}$ by the increase of $\lambda, 1/\mu$ and $\gamma$. Thus, there exist suitably large $\lambda$, $1/\mu$ and $\gamma$ such that for all $x(0) \in \mathcal{D}_{x}$, it has $x(i) \in \mathcal{D}_{x}, \forall t \geq 0$, where $\mathcal{D}_{x} \subset \mathcal{D}$. Consequently, the UUB result of all closed-loop signals holds for all $x(0) \in \mathcal{D}_{x}$.

(iii) Steady-state tracking performance. Combining (19) with (12) and Lemma 2, one also obtains

$$ \dot{V} \leq -k(t)(e_i - \bar{\xi})^2, \quad \forall[e_i] > \bar{\xi}_\mu $$

From the properties of the filtered tracking error $e_i$ (see Remark 2.1 of [27]), one concludes that the tracking error $e$ finally converges to a residual set

$$ \Omega_z := \{e_i|e_i| \leq 2\lambda^{i-1} \bar{\xi}_\mu, \quad i = 1, 2, \ldots, n, \quad y_d \in \Omega_d\} \quad (24) $$

that can be made arbitrarily small by the increase of $1/\mu$ and $\lambda$, which implies that the closed-loop system achieves practical asymptotic tracking performance [32].

**Remark 3:** The major challenge of this study is the stability analysis of the entire system under NN approximation and filtered adaptation. In our approach, two important state domains, that is, the domain of global stabilisation $\mathcal{D}_x$ and the domain of attraction $\mathcal{S}$, are exploited to resolve this challenge. In the proof of Theorem 2, it is shown that $\mathcal{D}_x$ in (13) satisfying $\mathcal{D}_x \subset \mathcal{D}_x$ can be covered by $\mathcal{S}$ in (23) (i.e. $\mathcal{S} \supseteq \mathcal{D}_x$) via a simple adjustment the control parameters $\lambda, \mu$ in (12) and $\gamma$ in (15), which guarantees semi-global practical asymptotic stability of the closed-loop system. Thus, high-frequency content can be effectively and safely filtered out by the adaptive laws in (14) and (15), so that fast and low-frequency adaptation can be obtained under high-gain learning rates. Please refer to Remark 3.2 of [24] for the detailed principle of fast and low-frequency adaptation.

**Remark 4:** Typically, trial and error is unavoidable for the control of uncertain non-linear systems. In Theorems 1 and 2, the control parameters $\alpha$ in (14) and $\sigma$ in (15) can be specified, the domain of global stabilisation $\mathcal{D}_x$ in (13) can be contracted by the increase of $\lambda$ and $1/\mu$ to guarantee $\mathcal{D}_x \subset \mathcal{D}_x$, the domain of attraction $\mathcal{S}$ in (23) can be enlarged by the increase of $\lambda$, $1/\mu$ and $\gamma$ to guarantee $\mathcal{S} \supseteq \mathcal{D}_x$, and the steady-state residual set $\Omega_i$ in (24) can be made arbitrarily small by the increase of $\lambda$ and $1/\mu$. Simply, the control performances in Theorems 1 and 2 can be guaranteed and further enhanced by the increase of $\lambda$, $1/\mu$ and $\gamma$, which provides a simple trial and error method for semi-global practical asymptotic tracking ANNC of the uncertain non-linear system in (1).

### 4 An illustrative example

To demonstrate the effectiveness of the proposed approach, consider an aircraft wing rock model as follows [24]

$$ \begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \alpha_1x_1 + \alpha_2 + \alpha_3|x_1|x_2 + \alpha_4|x_1| + x_2 + (1 + 0.2\sin(x_1))u \\
y &= x_1
\end{aligned} \quad (25) $$

where $x_1$ and $x_2$ are the position and velocity of wing rock, respectively, and $\alpha_i$ with $i = 1, 2, \ldots, 5$ are unknown parameters determined by the aircraft aerodynamic coefficients. For simulation, set $\alpha_1 = 0.2314$, $\alpha_2 = 0.7848$, $\alpha_3 = -0.0624$, $\alpha_4 = 0.0095$ and $\alpha_5 = 0.0215$. Differing from [24], the structure of the system in (25) is assumed to be unknown, $g(\cdot)$ is an unknown function, and $x(0)$ is outside of the approximation domain $\mathcal{D}_x$. The tracking indices $J(\text{ITAE})$ and $J(\text{IAE})$ with respect to $e_i$ and the control energy $E_c$, are chosen as control performance indices [7].

From the system description in the previous paragraph, the parameters of $\Omega_x, \Omega_d, \Omega_z, \mathcal{D}_x$ and $\mathcal{D}_y$ can be determined as $c_0 = 0.3$, $c_{de} = 0.8$, $c_\mu = 30$, $c_\gamma = 0.9$ and $c_\mu = 0.7$, respectively. The procedure of control design is as follows: firstly, select activation functions $\mu_i(x_i) = \exp(-(x_i - (\pi/16)(i - 2))i/(\pi/42)^2)$ with $x_i = [x_1, \ldots, x_6]^T$, $l_i = 1, 2, 3$ and $i = 1, \ldots, 4$ to construct the vector of basis functions $\Phi(x)$ in (6); secondly, select $\mu = 1$ and $\lambda = 1$ in (12) to determine the PD control $u_c$ in (7); third, choose $\vec{W}(0) = 0$, $\alpha_i \in [1, 5]^5$, $\sigma \in [0.1, 0.0001]$ and $\gamma \in [100, 10000]$ for the adaptive laws in (14) and (15).

**Case 1:** Normal control tasks. Let the initial state vector $x(0) = [\pi/12, 0]^T$ and the desired input $y_d = 2y_c/(x^2 + 3x + 2)$, where $y_c$ is a square command signal that belongs to $[-\pi/18, \pi/18]$ with the period 30s. The proposed approach is compared with its non-filtering counterpart. Control trajectories of two controllers under low-gain adaptation ($\gamma = 1 \times 10^2, \alpha_2 = 5$ and $\sigma = 1 \times 10^{-1}$) and high-gain adaptation ($\gamma = 1 \times 10^4, \alpha_2 = 125$ and $\sigma = 1 \times 10^{-5}$) are demonstrated in Figs. 1 and 2, respectively. It is observed that under the low-gain adaptation, both controllers achieve similar tracking performance, and the proposed controller obtains smoother control input $u$, whereas under the high-gain adaptation, both controllers achieve superb tracking performance with fast adaptation speed, the non-filtering controller generates serious chattering at both the filtered tracking error $e_i$ and the control input $u$, and the proposed controller achieves better tracking performance under smooth control input $u$ without chattering.

Performance comparison of two controllers under varying learning rates with $\alpha_i = 5^i$ and $\sigma = 1 \times 10^{-2}$ is shown in Table 1. It is observed that for both controllers, increasing the learning rate $\gamma$ improves tracking performance at the cost of high control energy (i.e. control cost or control efforts), and the proposed controller obviously outperforms its counterpart in terms of tracking accuracy and control cost under high-gain adaptation. To further indicate the efforts of $\alpha_i$ and $\sigma$, performance comparison of the proposed controller under varying $\alpha_i$ and $\sigma$ is given in Table 2. It is observed that increasing $\alpha_i$ or decreasing $\sigma$ increases control cost but does not always improve tracking performance. It seems that for a given control task, there exist optimal $\alpha_i$ and $\sigma$ to achieve the best tracking performance.

**Case 2:** Semi-global stabilisation. Let the initial state vector $x(0) = [\pi/3, 0]^T$ and the desired output $y_d$ be the same
as Case 1. Control trajectories of the proposed controller under high-gain adaptation ($\gamma = 1 \times 10^4$, $\alpha_f = 125$ and $\sigma = 1 \times 10^{-2}$) are depicted in Fig. 3. It is observed that the proposed controller successfully controls the plant with fast convergent speed and high tracking accuracy even $x(0)$ is extremely outside of the approximation domain $D_x$.

Case 3: Fast adaptation under disturbances. Let the initial state vector $x(0)$ be the same as Case 1, and the desired input $y_d = 25y_c/(s^2 + 10s + 25)$, where $y_c$ is a square command signal that belongs to $[-\pi/18, \pi/18]$ with the period 10 s. In addition, an external disturbance $d(t) = 0.2 \cos(10t)$ is added to the system in (25) after $t = 7$ s. Control trajectories of the proposed controller under high-gain adaptation ($\gamma = 1 \times 10^4$, $\alpha_f = 125$ and $\sigma = 1 \times 10^{-2}$) are displayed in Fig. 4. It is shown that the proposed controller still keeps favourable control performance with high tracking accuracy.
Table 1 Performance comparison of different controllers under varying learning rates

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<th>Performance indices</th>
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Table 2 Performance comparison of the proposed controller under varying control parameters

<table>
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<th>Performance indices</th>
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Fig. 3 Semi-global stabilisation by the proposed controller under extreme initial states

and smooth control input $u$ for the fast varying $y_d$ and the high-frequency disturbance $d(t)$. Note that the amplitudes of the $d(t)$, $f(x)$ and $y_d$ are comparable since $d(t) \leq 0.2$, $f(x) < 0.13$ and $y_d \leq \pi/18 < 0.18$.

Fig. 4 Fast adaptation by the proposed controller under external disturbances

5 Conclusions

This paper has successfully developed a novel ANC plus PD control strategy for a class of uncertain non-linear systems. The major contribution of this study is that semi-global practical asymptotic stability of the closed-loop system is established under the proposed control strategy, so that fast and low-frequency adaptation can be obtained via high-gain learning rates under guaranteed system stability. It is shown from simulation results that the proposed approach can safely improve tracking performance while keeping smooth control input by the increase of adaptation gain even in the presence of external disturbances, and can outperform its non-filtering counterpart in terms of tracking accuracy and energy cost in the face of high-gain adaptation. However, for the selection of the control parameters $\alpha$ and $\sigma$, there is a trade-off between tracking performance and control smoothness. Further work will focus on the determination of the optimal $\alpha$ and $\sigma$ to achieve the best control performance under the proposed control scheme.

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7 References


